A Generalised cut-elimination Procedure through Subatomic Proof Systems

Andrea Aler Tubella
(joint work with Alessio Guglielmi)

University of Bath

9th of April 2016

Deep inference web site: `http://alessio.guglielmi.name/res/cos/`

Outline

Introduction

Proof systems with a single rule

The proof system

Proofs, soundness and completeness

Examples

Cut-elimination in two phases

Splitting

Decomposition

Conclusion

Introduction

- We observe that deep inference systems have a recurring linear "rule shape". Can you see it?

$$\mathsf{ai}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{m}\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\mathsf{aw}{\downarrow}\,\frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{aw}{\uparrow}\,\frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\mathsf{ac}{\downarrow}\,\frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\,\frac{S\{a\}}{S(a,a)}$$

Figure: SKS [1]

$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{\bot\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}{\downarrow}\,\frac{S\llparenthesis [R,U],[T,V] \rrparenthesis}{S[\llparenthesis R,T \rrparenthesis, \lBrace U,V \rBrace]} \qquad \mathsf{d}{\uparrow}\,\frac{S(\lBrace R,U \rBrace, \llparenthesis T,V \rrparenthesis)}{S\lBrace (R,T),(U,V) \rBrace}$$

$$\mathsf{p}{\downarrow}\,\frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}{\uparrow}\,\frac{S(?R,!T)}{S\{?(R,T)\}}$$

$$\mathsf{at}{\downarrow}\,\frac{S\{0\}}{S\{a\}} \qquad \mathsf{ac}{\downarrow}\,\frac{S\lBrace a,a \rBrace}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\,\frac{S\{a\}}{S\llparenthesis a,a \rrparenthesis} \qquad \mathsf{at}{\uparrow}\,\frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{nm}{\downarrow}\,\frac{S\{0\}}{S\llparenthesis 0,0 \rrparenthesis} \qquad \mathsf{m}\,\frac{S\lBrace \llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis \rBrace}{S\llparenthesis \lBrace R,T \rBrace, \lBrace U,V \rBrace \rrparenthesis} \qquad \mathsf{nm}{\uparrow}\,\frac{S\lBrace \top,\top \rBrace}{S\{\top\}}$$

$$\mathsf{nm}_1{\downarrow}\,\frac{S\{0\}}{S[0,0]} \qquad \mathsf{m}_1{\downarrow}\,\frac{S\lBrace [R,U],[T,V] \rBrace}{S[\lBrace R,T \rBrace, \lBrace U,V \rBrace]} \qquad \mathsf{m}_1{\uparrow}\,\frac{S(\llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis)}{S\llparenthesis (R,T),(U,V) \rrparenthesis} \qquad \mathsf{nm}_1{\uparrow}\,\frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{nm}_2{\downarrow}\,\frac{S\{0\}}{S(0,0)} \qquad \mathsf{m}_2{\downarrow}\,\frac{S\lBrace (R,U),(T,V) \rBrace}{S(\lBrace R,T \rBrace, \lBrace U,V \rBrace)} \qquad \mathsf{m}_2{\uparrow}\,\frac{S[\llparenthesis R,U \rrparenthesis, \llparenthesis T,V \rrparenthesis]}{S\llparenthesis [R,T],[U,V] \rrparenthesis} \qquad \mathsf{nm}_2{\uparrow}\,\frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{nl}_1{\downarrow}\,\frac{S\{0\}}{S\{?0\}} \qquad \mathsf{l}_1{\downarrow}\,\frac{S\lBrace ?R,?T \rBrace}{S\{?\lBrace R,T \rBrace\}} \qquad \mathsf{l}_1{\uparrow}\,\frac{S\{!\llparenthesis R,T \rrparenthesis\}}{S\llparenthesis !R,!T \rrparenthesis} \qquad \mathsf{nl}_1{\uparrow}\,\frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{nl}_2{\downarrow}\,\frac{S\{0\}}{S\{!0\}} \qquad \mathsf{l}_2{\downarrow}\,\frac{S\lBrace !R,!T \rBrace}{S\{!\lBrace R,T \rBrace\}} \qquad \mathsf{l}_2{\uparrow}\,\frac{S\{?\llparenthesis R,T \rrparenthesis\}}{S\llparenthesis ?R,?T \rrparenthesis} \qquad \mathsf{nl}_2{\uparrow}\,\frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{nz}{\downarrow}\,\frac{S\{\bot\}}{S\{?0\}} \qquad \mathsf{z}{\downarrow}\,\frac{S[?R,T]}{S\{?\lBrace R,T \rBrace\}} \qquad \mathsf{z}{\uparrow}\,\frac{S\{!\llparenthesis R,T \rrparenthesis\}}{S(!R,T)} \qquad \mathsf{nz}{\uparrow}\,\frac{S\{!\top\}}{S\{1\}}$$

Figure: SLLS [4]

Introduction

- We observe that deep inference systems have a recurring linear "rule shape". Can you see it?

$$\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\boxed{\mathsf{m}\, \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}}$$

$$\mathsf{aw}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{aw}\!\uparrow \frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\mathsf{ac}\!\downarrow \frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}\!\uparrow \frac{S\{a\}}{S(a,a)}$$

Figure: SKS [1]

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\bot\}}$$

$$\mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\boxed{\mathsf{d}\!\downarrow \frac{S\langle [R,U],[T,V]\rangle}{S[\langle R,T\rangle,\{\!\!\{U,V\}\!\!\}]}} \qquad \boxed{\mathsf{d}\!\uparrow \frac{S(\{\!\!\{R,U\}\!\!\},\langle T,V\rangle)}{S\{\!\!\{(R,T),(U,V)\}\!\!\}}}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}$$

$$\mathsf{at}\!\downarrow \frac{S\{0\}}{S\{a\}} \qquad \mathsf{ac}\!\downarrow \frac{S\{\!\!\{a,a\}\!\!\}}{S\{a\}} \qquad \mathsf{ac}\!\uparrow \frac{S\{a\}}{S\langle a,a\rangle} \qquad \mathsf{at}\!\uparrow \frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{nm}\!\downarrow \frac{S\{0\}}{S\langle 0,0\rangle} \qquad \boxed{\mathsf{m}\, \frac{S\{\!\!\{\langle R,U\rangle,\langle T,V\rangle\}\!\!\}}{S\langle \{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\}\rangle}} \qquad \mathsf{nm}\!\uparrow \frac{S\{\!\!\{\top,\top\}\!\!\}}{S\{\top\}}$$

$$\mathsf{nm_1}\!\downarrow \frac{S\{0\}}{S[0,0]} \qquad \boxed{\mathsf{m_1}\!\downarrow \frac{S\{\!\!\{[R,U],[T,V]\}\!\!\}}{S[\{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\}]}} \qquad \boxed{\mathsf{m_1}\!\uparrow \frac{S(\langle R,U\rangle,\langle T,V\rangle)}{S\langle (R,T),(U,V)\rangle}} \qquad \mathsf{nm_1}\!\uparrow \frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{nm_2}\!\downarrow \frac{S\{0\}}{S(0,0)} \qquad \boxed{\mathsf{m_2}\!\downarrow \frac{S\{\!\!\{(R,U),(T,V)\}\!\!\}}{S(\{\!\!\{R,T\}\!\!\},\{\!\!\{U,V\}\!\!\})}} \qquad \boxed{\mathsf{m_2}\!\uparrow \frac{S[\langle R,U\rangle,\langle T,V\rangle]}{S\langle [R,T],[U,V]\rangle}} \qquad \mathsf{nm_2}\!\uparrow \frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{nl_1}\!\downarrow \frac{S\{0\}}{S\{?0\}} \qquad \mathsf{l_1}\!\downarrow \frac{S\{\!\!\{?R,?T\}\!\!\}}{S\{?\{\!\!\{R,T\}\!\!\}\}} \qquad \mathsf{l_1}\!\uparrow \frac{S\{!\langle R,T\rangle\}}{S\langle !R,!T\rangle} \qquad \mathsf{nl_1}\!\uparrow \frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{nl_2}\!\downarrow \frac{S\{0\}}{S\{!0\}} \qquad \mathsf{l_2}\!\downarrow \frac{S\{\!\!\{!R,!T\}\!\!\}}{S\{!\{\!\!\{R,T\}\!\!\}\}} \qquad \mathsf{l_2}\!\uparrow \frac{S\{?\langle R,T\rangle\}}{S\langle ?R,?T\rangle} \qquad \mathsf{nl_2}\!\uparrow \frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{nz}\!\downarrow \frac{S\{\bot\}}{S\{?0\}} \qquad \mathsf{z}\!\downarrow \frac{S[?R,T]}{S\{?\{\!\!\{R,T\}\!\!\}\}} \qquad \mathsf{z}\!\uparrow \frac{S\{!\langle R,T\rangle\}}{S(!R,T)} \qquad \mathsf{nz}\!\uparrow \frac{S\{!\top\}}{S\{1\}}$$

Figure: SLLS [4]

Proof systems with a single rule scheme

- Goal: generating propositional proofs by a single, linear, simple and regular inference rule scheme.

  So how do we change the atomic rules to fit our scheme?
- Idea: consider atoms as self-dual, noncommutative binary logical relations.
  - extended language of formulae (atoms inside atoms, whatever that means)
- Why: We can use the rule scheme to reason very generally on proof systems.
  - general cut-elimination procedure for many systems without contraction, including all the standard variants of linear logic

The proof system

We use propositional classical logic as an example.

Idea: occurrences of an atom *a* are interpretations of more primitive expressions involving a noncommutative binary relation denoted by *a*.

- Formulae *A* and *B* in the relation *a*, in this order, are denoted by $A\ a\ B$.
- Formulae are built over the two units for disjunction and conjunction, respectively 0 and 1.

Example: the following two expressions are SA formulae:

$$\langle 0\ a\ 1\rangle \vee \langle 1\ a\ 0\rangle \qquad \langle 0\ b\ 1\rangle\ a\ \langle 1\ c\ \langle 1\ d\ 0\rangle\rangle \wedge 0 \wedge [\langle 0\ a\ 0\rangle \vee \langle 1\ b\ 1\rangle]$$

We call tame the formulae where atoms do not appear in the scope of other atoms (e.g., left) and wild the others (e.g., right).

The proof system

To interpret our extended language of formulae, we define an interpretation map $\mapsto$ from tame SA formulae to ordinary formulae such that

$$0\,a\,1 \mapsto a \quad 1\,a\,0 \mapsto \bar{a}$$
$$1\,a\,1 \mapsto \mathsf{t} \quad 0\,a\,0 \mapsto \mathsf{f}$$

where $\bar{a}$ denotes the negation of a.

Note

- atoms are self dual: $\overline{A\,a\,B} \equiv \bar{A}\,a\,\bar{B}$
- atoms are not commutative
- atoms are not associative

We easily extend $\mapsto$ to all the tame SA formulae in the natural way. For example: $\langle 0\,a\,1\rangle \vee \langle 1\,a\,0\rangle \mapsto a \vee \bar{a}$ $\qquad [0 \vee 0]\,a\,[1 \vee 1] \mapsto a$

The proof system

Consider the usual contraction rule for an atom:

$$\frac{a \vee a}{a}$$

We could obtain this rule via $\mapsto$ as follows:

$$\frac{\langle 0\ a\ 1\rangle \vee \langle 0\ a\ 1\rangle}{[0 \vee 0]\ a\ [1 \vee 1]} \mapsto \frac{a \vee a}{a} \qquad \text{and} \qquad \frac{\langle 1\ a\ 0\rangle \vee \langle 1\ a\ 0\rangle}{[1 \vee 1]\ a\ [0 \vee 0]} \mapsto \frac{\bar{a} \vee \bar{a}}{\bar{a}} \quad .$$

We might consider those rules as generated by the linear scheme

$$\frac{\langle A\ a\ C\rangle \vee \langle B\ a\ D\rangle}{[A \vee B]\ a\ [C \vee D]}$$

This scheme is typical of logical rules in deep inference.

The proof system

Two more examples, identity and cut:

$$\frac{[0 \vee 1]\ a\ [1 \vee 0]}{\langle 0\ a\ 1\rangle \vee \langle 1\ a\ 0\rangle} \mapsto \frac{\mathsf{t}}{a \vee \bar{a}} \quad \text{and} \quad \frac{\langle 0\ a\ 1\rangle \wedge \langle 1\ a\ 0\rangle}{(0 \wedge 1)\ a\ (1 \wedge 0)} \mapsto \frac{a \wedge \bar{a}}{\mathsf{f}} \quad .$$

They are generated by the linear schemes:

$$\frac{[A \vee C]\ a\ [B \vee D]}{\langle A\ a\ B\rangle \vee \langle C\ a\ D\rangle} \quad \text{and} \quad \frac{\langle A\ a\ C\rangle \wedge \langle B\ a\ D\rangle}{(A \wedge B)\ a\ (C \wedge D)}$$

- Surprisingly, we are able to reduce disparate rules such as contraction, cut and identity into a unique rule scheme.

The proof system

- A subatomic system SA is a deep inference system whose rules are instances of the inference rule scheme

$$\frac{(A \, \alpha \, C) \; \beta \; (B \, \gamma \, D)}{(A \, \beta \, B) \; \alpha \; (C \, \delta \, D)}$$

where A, B, C, D are formulae and $\alpha, \beta, \gamma, \delta$ are relations.

Proofs, soundness and completeness

- What is a proof in a SA system? A proof is a derivation whose premiss is equal to a distinguished unit u.

  Example of Classical Logic:
  A proof is a derivation whose premiss is equal to t.
- A proof composed of only tame formulae corresponds to a proof in our usual proof theory.

$$
\dfrac{\dfrac{[0 \vee 1]\; a\; [1 \vee 0]}{\langle 0\; a\; 1 \rangle \vee \langle 1\; a\; 0 \rangle} \wedge \dfrac{[1 \vee 0]\; a\; [0 \vee 1]}{\langle 1\; a\; 0 \rangle \vee \langle 0\; a\; 1 \rangle}}{\dfrac{\langle 0\; a\; 1 \rangle \wedge \langle 1\; a\; 0 \rangle}{(0 \wedge 1)\; a\; (1 \wedge 0)} \vee [\langle 1\; a\; 0 \rangle \vee \langle 0\; a\; 1 \rangle]} \quad \mapsto \quad \dfrac{\dfrac{\mathsf{t}}{a \vee \bar{a}} \vee \dfrac{\mathsf{t}}{\bar{a} \vee a}}{\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee [\bar{a} \vee a]}
$$

Proofs, soundness and completeness

We can define the soundness and completeness of a subatomic system by using the interpretation function.

- Soundness: check that each inference rule instance involving tame formulae corresponds to a valid implication between premiss and conclusion.
- Completeness: show that each inference rule (on tame formulae) of a complete system for propositional logic, such as KS, can be represented by one or more rules of SA.

We have sound and complete SA systems corresponding to a big variety of deep inference systems.

Examples

$$\mathsf{ai}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{m}\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\mathsf{aw}{\downarrow}\,\frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{aw}{\uparrow}\,\frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\mathsf{ac}{\downarrow}\,\frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\,\frac{S\{a\}}{S(a,a)}$$

Figure: SKS [1]

$$i{\downarrow}\,\frac{[A \vee B]\; a\; [C \vee D]}{\langle A\; a\; C\rangle \vee \langle B\; a\; D\rangle} \qquad i{\uparrow}\,\frac{\langle A\; a\; B\rangle \wedge \langle C\; a\; D\rangle}{(A \wedge C)\; a\; (B \wedge D)}$$

$$\frac{[A \vee B] \wedge [C \vee D]}{(A \wedge C) \vee [B \vee D]}\, s$$

$$\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}\, m$$

$$c{\downarrow}\,\frac{\langle A\; a\; B\rangle \vee \langle C\; a\; D\rangle}{[A \vee C]\; a\; [B \vee D]} \qquad c{\uparrow}\,\frac{(A \wedge B)\; a\; (C \wedge D)}{\langle A\; a\; C\rangle \wedge \langle B\; a\; D\rangle}$$

Figure: SAKS

Examples

$$\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\circ\}}$$

$$\mathsf{q}\!\downarrow \frac{S\langle [R,T];[R',T']\rangle}{S[\langle R;R'\rangle,\langle T;T'\rangle]} \qquad \mathsf{q}\!\uparrow \frac{S(\langle R;T\rangle,\langle R';T'\rangle)}{S\langle (R,R');(T,T')\rangle}$$

$$\mathsf{s} \frac{S([R,T],R')}{S[(R,R'),T]}$$

Figure: SBV [3]

$$i\!\downarrow \frac{[A\ ,\ B]\ a\ [C\ ,\ D]}{[\langle A\ a\ C\rangle\ ,\ \langle B\ a\ D\rangle]} \qquad i\!\uparrow \frac{(\langle A\ a\ B\rangle\ ,\ \langle C\ a\ D\rangle)}{(A\ ,\ C)\ a\ (B\ ,\ D)}$$

$$q\!\downarrow \frac{\langle [A\ ,\ B];[C\ ,\ D]\rangle}{[\langle A\ ;\ C\rangle\ ,\ \langle B\ ;\ D\rangle]} \qquad q\!\uparrow \frac{(\langle A\ ;\ B\rangle\ ,\ \langle C\ ;\ D\rangle)}{\langle (A\ ,\ C)\ ;\ (B\ ,\ D)\rangle}$$

$$s \frac{([A\ ,\ B];[C\ ,\ D])}{[(A\ ,\ C),[B\ ,\ D]}$$

Figure: SABV

Cut-elimination in two phases

We break down cut-elimination into two different steps:

- Splitting: a cut-elimination procedure for systems without contractions.
- Decomposition: in other systems, we decompose proofs into a top phase without contractions, and a bottom phase made-up only of contractions.

Splitting

- Idea: Being able to check whether a system has cut-elimination just by checking a few things.
- We provide a generalised cut-elimination procedure that can be applied to the logics that do not have contractions.
- The method is a generalisation of splitting [3].
- Splitting has been shown to work in all deep inference systems studied so far, and we gain insight as to *why*.

Splitting

Theorem (Splitting)

For every proof $\begin{array}{c}\mathsf{t}\\ \| \\ K\{A \wedge B\}\end{array}$ *there are derivations*

$$\begin{array}{c}K_A \vee K_B \vee \{\ \}\\ \| \\ K\{\ \}\end{array} \qquad \begin{array}{c}\mathsf{t}\\ \| \\ K_A \vee A\end{array} \qquad \begin{array}{c}\mathsf{t}\\ \| \\ K_B \vee B\end{array} \quad .$$

An alike theorem holds for every logic expressed in deep inference so far.

Splitting

Therefore for every cut-free proof
$$\begin{array}{c} \mathsf{t} \\ \Vert \\ K\{a \wedge \bar{a}\} \end{array}$$

there are cut-free proofs

$$\begin{array}{c} K_a \vee K_{\bar{a}} \vee \{\ \} \\ \Vert \\ K\{\ \} \end{array} \qquad \begin{array}{c} \mathsf{t} \\ \Vert \\ K_a \vee a \end{array} \qquad \begin{array}{c} \mathsf{t} \\ \Vert \\ K_{\bar{a}} \vee \bar{a} \end{array}$$

and so we can build
$$\begin{array}{c} \mathsf{t} \\ \mathsf{i}\!\downarrow \overline{} \\ \boxed{\begin{array}{c} a \\ \Vert \\ K_{\bar{a}} \end{array} \vee \begin{array}{c} \bar{a} \\ \Vert \\ K_a \end{array}} \\ \Vert \\ K\{\mathsf{f}\} \end{array}$$

and a cut at the bottom would be admissible.

Splitting

- The main idea is to follow the relations involved in the cut from the bottom to the top to find where they are introduced.
- We need these relations to be non-contractive: their scope only widens from the bottom to the top of a proof.

In the subatomic contraction rule itself we can see that the scope of the atom narrows:

$$\frac{\langle A\ a\ C \rangle \vee \langle B\ a\ D \rangle}{[A \vee B]\ a\ [C \vee D]}$$

These are the type of rules that we do not have in a splittable system, the subatomic equivalent to a linear system.

Splitting

Theorem

If SA *is a splittable system, then for every proof* $\begin{array}{c} u \\ \| \\ K\{A\ \alpha\ B\} \end{array}$ *where* $\alpha \neq \vee$*,*

there are derivations

$$\begin{array}{c} (K_A\ \overline{\alpha}\ K_B) \vee \{\ \} \\ \| \\ K\{\ \} \end{array} \qquad \begin{array}{c} u \\ \| \\ K_A \vee A \end{array} \qquad \begin{array}{c} u \\ \| \\ K_B \vee B \end{array} \qquad .$$

Splitting

Corollary

Given

$$K\left\{\begin{array}{c} u \\ {\scriptstyle \mathsf{SA}}\Vert \\ \rho\,\dfrac{(A\,\alpha\,C)\,\beta\,(B\,\alpha\,D)}{(A\,\beta\,B)\,\alpha\,(C\,\beta\,D)} \end{array}\right\}$$

there is a proof

$$\begin{array}{c} u \\ {\scriptstyle \mathsf{SA}\cup\bar{\rho}}\Vert \\ K\{(A\,\beta\,B)\,\alpha\,(C\,\beta\,D)\} \end{array} \quad .$$

Splitting

Proof.

By applying Splitting twice, there are K_A, K_B, K_C, K_D such that

$$\dfrac{\dfrac{\boxed{\begin{array}{c} u \\ \| \\ K_A \vee A \end{array}} \;\beta\; \boxed{\begin{array}{c} u \\ \| \\ K_C \vee C \end{array}}}{(K_A \,\overline{\beta}\, K_C) \vee (A \,\beta\, C)} \quad \alpha \quad \dfrac{\boxed{\begin{array}{c} u \\ \| \\ K_B \vee B \end{array}} \;\beta\; \boxed{\begin{array}{c} u \\ \| \\ K_D \vee D \end{array}}}{(K_B \,\overline{\beta}\, K_D) \vee (B \,\beta\, D)}}{\bar{\rho}\; \dfrac{(K_A \,\overline{\beta}\, K_C) \,\overline{\alpha}\, (K_B \,\overline{\beta}\, K_D)}{\boxed{\begin{array}{c} \boxed{\begin{array}{c} K_A \,\overline{\alpha}\, K_B \\ \| \\ K_1 \end{array}} \;\overline{\beta}\; \boxed{\begin{array}{c} K_C \,\overline{\alpha}\, K_D \\ \| \\ K_2 \end{array}} \\ \| \\ K \end{array}}} \vee ((A \,\beta\, B) \,\alpha\, (C \,\beta\, D))}$$

Decomposition

- In many systems, derivations can be arranged into consecutive subderivations made up of only certain rules: we call this transformation decomposition.
- Herbrand's Theorem (Ben's talk) is an example: bottom phase with contraction and quantifier rules and a top phase with propositional rules only.
- If we can decompose a sytem into a top phase with no contraction rules and a bottom phase with all the contraction rules, we can apply splitting to the top and eliminate cuts.

Decomposition

- In particular, for system SAKS for Classical Logic we can decompose proofs into a non-contractive phase followed by a phase made-up of only contractive rules.
- Interestingly, we can provide a decomposition procedure where all the steps are local, meaning that we can do it by rewriting specific subderivations in the proof without having to consider the proof as a whole.

Conclusion

- We observe a mysterious phenomenon: only one rule shape is enough to describe many different systems.
- Exploiting this unique shape, we can generalise procedures such as splitting.
- In the process, we learn about cut-elimination: we can separate into two steps that are usually intertwined.
- For example, we can study which parts of cut-elimination are global and which are local.
- We are able to observe that complexity comes from decomposition rather than from splitting.
- We would like to use it as a stepping stone towards a geometrical formalism.

References

[1] K. Brünnler and A. F. Tiu.
A local system for classical logic.
In R. Nieuwenhuis and A. Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, volume 2250 of *Lecture Notes in Computer Science*, pages 347–361. Springer-Verlag, 2001.

[2] A. Guglielmi.
Some news on subatomic logic, 2005.

[3] A. Guglielmi.
A system of interaction and structure.
ACM Transactions on Computational Logic, 8(1):1:1–64, 2007.

[4] L. Straßburger.
Linear Logic and Noncommutativity in the Calculus of Structures.
PhD thesis, Technische Universität Dresden, 2003.