Generalising cut elimination through subatomic proof systems

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Deep inference web site: `http://alessio.guglielmi.name/res/cos/`

Objectives

- Understanding why cut-elimination works.
- Characterising proof-systems whose proofs can be normalised.
- Developing a geometrical representation of proofs to solve the problem of the identity of proofs (Hilbert's 24th problem).

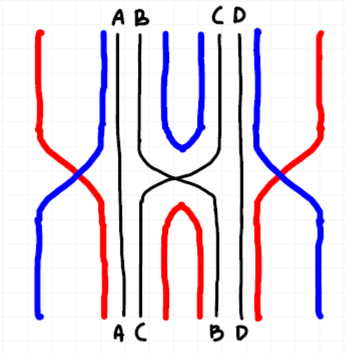
One shape to rule them all

- Many proof systems can be represented in such a way that every inference rule is an instance of a single linear inference rule scheme.

Syntax:

$$\frac{(A \; \alpha \; B) \; \beta \; (C \; \alpha' \; D)}{(A \; \beta \; C) \; \alpha \; (B \; \beta' \; D)}$$

Topological intuition via ribbons:



But how?

- This shape arises very often when we have atomic introduction and contraction rules.

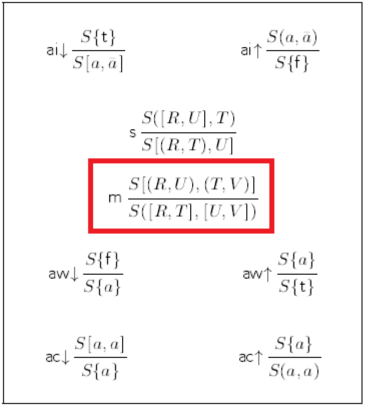


Figure: SKS [1]

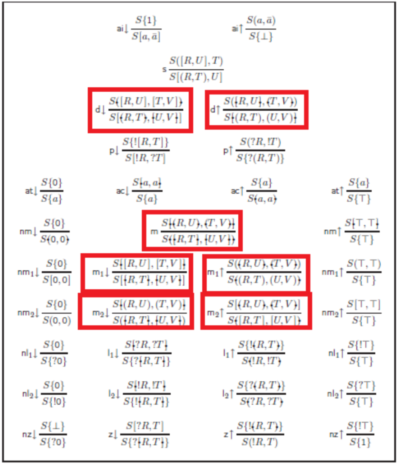


Figure: SLLS [4]

But how?

- How do the atomic rules fit the scheme?

We can consider atoms as superpositions of truth values:

$$0 \; a \; 1 \approx a \qquad 1 \; a \; 0 \approx \bar{a}$$
$$1 \; a \; 1 \approx 1 \qquad 0 \; a \; 0 \approx 0$$

- How does that change the rules?

Contraction:

$$\frac{(A \; a \; B) \vee (C \; a \; D)}{(A \vee C) \; a \; (B \vee D)}$$

But how?

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- How does that change the rules?

Contraction:

$$\frac{(0\ a\ 1) \vee (0\ a\ 1)}{(0 \vee 0)\ a\ (1 \vee 1)} \approx \frac{a \vee a}{a}$$

But how?

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We can consider atoms as superpositions of truth values:

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$$1 \; a \; 1 \approx 1 \qquad 0 \; a \; 0 \approx 0$$

- How does that change the rules?

Contraction:

$$\frac{(1 \; a \; 0) \vee (1 \; a \; 0)}{(1 \vee 1) \; a \; (0 \vee 0)} \approx \frac{\bar{a} \vee \bar{a}}{\bar{a}}$$

But how?

Two more examples, identity and cut:

$$\frac{(0 \vee 1) \; a \; (1 \vee 0)}{(0 \; a \; 1) \vee (1 \; a \; 0)} \approx \frac{1}{a \vee \bar{a}} \qquad \text{and} \qquad \frac{(0 \; a \; 1) \wedge (1 \; a \; 0)}{(0 \wedge 1) \; a \; (1 \wedge 0)} \mapsto \frac{a \wedge \bar{a}}{0} \quad .$$

They are generated by the linear schemes:

$$\frac{(A \vee C) \; a \; (B \vee D)}{(A \; a \; B) \vee (C \; a \; D)} \qquad \text{and} \qquad \frac{(A \; a \; C) \wedge (B \; a \; D)}{(A \wedge B) \; a \; (C \wedge D)}$$

- Surprisingly, we are able to reduce disparate rules such as contraction, cut and identity into a unique rule scheme.

But how?

- Can we make proof systems for that?
  - Not in Gentzen formalisms.
  - Yes in Deep Inference.

- Deep Inference is necessary for complete proof systems with self-dual non-commutative connectives[5].

Results

- We can represent a large class of logics, from Classical Logic to MELL.
- We can provide a generalised cut-elimination procedure (Splitting) for a large class of linear logics, that we can easily characterise.
- We can extend splitting to non-linear logics that do not form loops (decomposition).
- We have some evidence that we can remove the restriction on loops.

Representing Logics

$$\text{ai}\downarrow\frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \text{ai}\uparrow\frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\text{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\text{m}\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\text{aw}\downarrow\frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \text{aw}\uparrow\frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\text{ac}\downarrow\frac{S[a,a]}{S\{a\}} \qquad \text{ac}\uparrow\frac{S\{a\}}{S(a,a)}$$

Figure: SKS [1]

$$i\downarrow\frac{[A \vee B]\; a\; [C \vee D]}{\langle A\; a\; C\rangle \vee \langle B\; a\; D\rangle} \qquad i\uparrow\frac{\langle A\; a\; B\rangle \wedge \langle C\; a\; D\rangle}{(A \wedge C)\; a\; (B \wedge D)}$$

$$\frac{[A \vee B] \wedge [C \vee D]}{(A \wedge C) \vee [B \vee D]}\,s$$

$$\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}\,m$$

$$c\downarrow\frac{\langle A\; a\; B\rangle \vee \langle C\; a\; D\rangle}{[A \vee C]\; a\; [B \vee D]} \qquad c\uparrow\frac{(A \wedge B)\; a\; (C \wedge D)}{\langle A\; a\; C\rangle \wedge \langle B\; a\; D\rangle}$$

Figure: SAKS

Representing Logics

$$\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\circ\}}$$

$$\mathsf{q}\!\downarrow \frac{S\langle [R,T];[R',T']\rangle}{S[\langle R;R'\rangle,\langle T;T'\rangle]} \qquad \mathsf{q}\!\uparrow \frac{S(\langle R;T\rangle,\langle R';T'\rangle)}{S\langle (R,R');(T,T')\rangle}$$

$$\mathsf{s}\frac{S([R,T],R')}{S[(R,R'),T]}$$

Figure: SBV [3]

$$i\!\downarrow \frac{[A\ ,\ B]\ a\ [C\ ,\ D]}{[\langle A\ a\ C\rangle\ ,\ \langle B\ a\ D\rangle]} \qquad i\!\uparrow \frac{(\langle A\ a\ B\rangle\ ,\ \langle C\ a\ D\rangle)}{(A\ ,\ C)\ a\ (B\ ,\ D)}$$

$$q\!\downarrow \frac{\langle [A\ ,\ B];[C\ ,\ D]\rangle}{[\langle A\ ;\ C\rangle\ ,\ \langle B\ ;\ D\rangle]} \qquad q\!\uparrow \frac{(\langle A\ ;\ B\rangle\ ,\ \langle C\ ;\ D\rangle)}{\langle (A\ ,\ C)\ ;\ (B\ ,\ D)\rangle}$$

$$s\frac{([A\ ,\ B];[C\ ,\ D])}{[(A\ ,\ C),[B\ ,\ D]}$$

Figure: SABV

Representing Logics

$$\mathsf{ai}\!\downarrow\frac{S\{1\}}{S[a,\bar{a}]}\qquad\mathsf{ai}\!\uparrow\frac{S(a,\bar{a})}{S\{\bot\}}$$

$$\mathsf{s}\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}\!\downarrow\frac{S\llparenthesis[R,U],[T,V]\rrparenthesis}{S[\llparenthesis R,T\rrparenthesis,\lBrace U,V\rBrace]}\qquad\mathsf{d}\!\uparrow\frac{S(\lBrace R,U\rBrace,\llparenthesis T,V\rrparenthesis)}{S\lBrace(R,T),(U,V)\rBrace}$$

$$\mathsf{p}\!\downarrow\frac{S\{![R,T]\}}{S[!R,?T]}\qquad\mathsf{p}\!\uparrow\frac{S(?R,!T)}{S\{?(R,T)\}}$$

$$\mathsf{at}\!\downarrow\frac{S\{0\}}{S\{a\}}\qquad\mathsf{ac}\!\downarrow\frac{S\lBrace a,a\rBrace}{S\{a\}}\qquad\mathsf{ac}\!\uparrow\frac{S\{a\}}{S\llparenthesis a,a\rrparenthesis}\qquad\mathsf{at}\!\uparrow\frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{nm}\!\downarrow\frac{S\{0\}}{S\llparenthesis 0,0\rrparenthesis}\qquad\mathsf{m}\frac{S\lBrace\llparenthesis R,U\rrparenthesis,\llparenthesis T,V\rrparenthesis\rBrace}{S\llparenthesis\lBrace R,T\rBrace,\lBrace U,V\rBrace\rrparenthesis}\qquad\mathsf{nm}\!\uparrow\frac{S\lBrace\top,\top\rBrace}{S\{\top\}}$$

$$\mathsf{nm}_1\!\downarrow\frac{S\{0\}}{S[0,0]}\qquad\mathsf{m}_1\!\downarrow\frac{S\lBrace[R,U],[T,V]\rBrace}{S[\lBrace R,T\rBrace,\lBrace U,V\rBrace]}\qquad\mathsf{m}_1\!\uparrow\frac{S(\llparenthesis R,U\rrparenthesis,\llparenthesis T,V\rrparenthesis)}{S\llparenthesis(R,T),(U,V)\rrparenthesis}\qquad\mathsf{nm}_1\!\uparrow\frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{nm}_2\!\downarrow\frac{S\{0\}}{S(0,0)}\qquad\mathsf{m}_2\!\downarrow\frac{S\lBrace(R,U),(T,V)\rBrace}{S(\lBrace R,T\rBrace,\lBrace U,V\rBrace)}\qquad\mathsf{m}_2\!\uparrow\frac{S[\llparenthesis R,U\rrparenthesis,\llparenthesis T,V\rrparenthesis]}{S\llparenthesis[R,T],[U,V]\rrparenthesis}\qquad\mathsf{nm}_2\!\uparrow\frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{nl}_1\!\downarrow\frac{S\{0\}}{S\{?0\}}\qquad\mathsf{l}_1\!\downarrow\frac{S\lBrace ?R,?T\rBrace}{S\{?\lBrace R,T\rBrace\}}\qquad\mathsf{l}_1\!\uparrow\frac{S\{!\llparenthesis R,T\rrparenthesis\}}{S\llparenthesis !R,!T\rrparenthesis}\qquad\mathsf{nl}_1\!\uparrow\frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{nl}_2\!\downarrow\frac{S\{0\}}{S\{!0\}}\qquad\mathsf{l}_2\!\downarrow\frac{S\lBrace !R,!T\rBrace}{S\{!\lBrace R,T\rBrace\}}\qquad\mathsf{l}_2\!\uparrow\frac{S\{?\llparenthesis R,T\rrparenthesis\}}{S\llparenthesis ?R,?T\rrparenthesis}\qquad\mathsf{nl}_2\!\uparrow\frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{nz}\!\downarrow\frac{S\{\bot\}}{S\{?0\}}\qquad\mathsf{z}\!\downarrow\frac{S[?R,T]}{S\{?\lBrace R,T\rBrace\}}\qquad\mathsf{z}\!\uparrow\frac{S\{!\llparenthesis R,T\rrparenthesis\}}{S(!R,T)}\qquad\mathsf{nz}\!\uparrow\frac{S\{!\top\}}{S\{1\}}$$

Figure: SLLS [4]

Splitting

- The main idea is to follow the relations involved in the cut from the bottom to the top to find where they are introduced.

- We need these relations to be non-contractive: their scope only widens from the bottom to the top of a proof.

In the subatomic contraction rule itself we can see that the scope of the atom narrows:

$$\frac{\langle A\ a\ C\rangle \vee \langle B\ a\ D\rangle}{[A \vee B]\ a\ [C \vee D]}$$

These are the type of rules that we do not have in a splittable system, the subatomic equivalent to a linear system.

Splitting

Theorem

If SA *is a splittable system, then for every proof* $\begin{array}{c}\Vert\\ K\{A\,\alpha\,B\}\end{array}$ *, there are derivations*

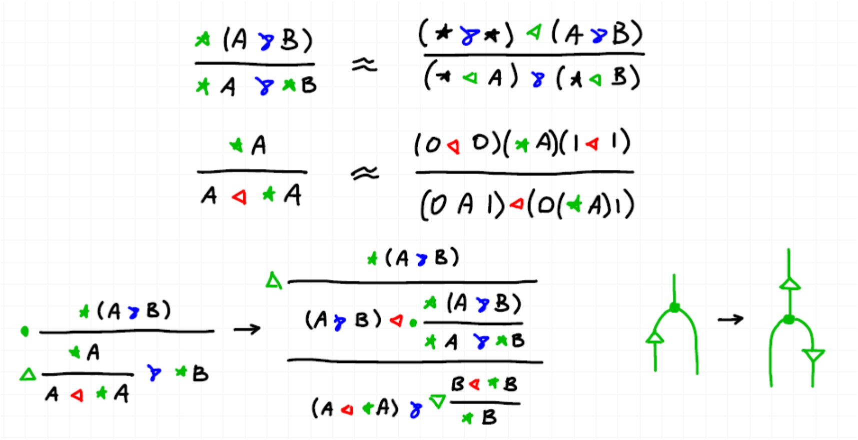
$$\begin{array}{c}(K_A\,\overline{\alpha}\,K_B)\vee\{\ \}\\ \Vert\\ K\{\ \}\end{array}\qquad \begin{array}{c}\Vert\\ K_A\vee A\end{array}\qquad \begin{array}{c}\Vert\\ K_B\vee B\end{array}\quad .$$

Decomposition

- In many systems, derivations can be arranged into consecutive subderivations made up of only certain rules: we call this transformation decomposition.
- Herbrand's Theorem (Ben's talk) is an example: bottom phase with contraction and quantifier rules and a top phase with propositional rules only.

Decomposition

- We have seen another example in Alessio's talk:



Decomposition

- We move contractions until we obtain a canonical proof. We can then apply splitting to the linear part of the proof.
- We can do decomposition on systems without loops, Classical Logic included.
- Decomposition is obtained through local transformations, as opposed to splitting that is a global procedure.

Conclusions

- We observe a striking phenomenon: only one rule shape is enough to describe many different systems.
- We can distinguish which parts of cut-elimination are global and which are local.
- We are able to observe that complexity comes from decomposition rather than from splitting.
- We would like to use it as a stepping stone towards a geometrical formalism.

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