Generalising Cut-Elimination through Subatomic Proof Systems
or
The Math Stuff That I Do

Andrea Aler Tubella

University of Bath

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- Curry-Howard isomorphism: programs and proofs are the same
  - the type returned is what we prove
  - the types of the arguments taken are assumptions
  - the program is a proof from the assumptions to the conclusion
- Feel free to substitute proof by program in all of my talk.

What is Mathematical Logic?

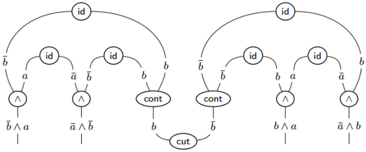
- We start from some very philosophical questions...
  - *What* can we prove?
  - What *is* a valid proof?
  - How *long* are the proofs?
- To answer them, in the best tradition of mathematics, we use symbolic languages to represent reasoning.

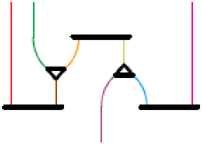
Representing proofs

Some examples:

$$\mathsf{cut}\,\dfrac{\mathsf{cont}\,\dfrac{\wedge\,\dfrac{\wedge\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash \bar b, b} \quad \mathsf{id}\,\dfrac{}{\vdash a, \bar a}}{\vdash \bar b \wedge a, \bar a, b} \quad \mathsf{id}\,\dfrac{}{\vdash \bar b, b}}{\vdash \bar b \wedge a, \bar a \wedge \bar b, b, b}}{\vdash \bar b \wedge a, \bar a \wedge \bar b, b} \qquad \mathsf{cont}\,\dfrac{\wedge\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash \bar b, b} \quad \wedge\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash a, \bar a} \quad \mathsf{id}\,\dfrac{}{\vdash \bar b, b}}{\vdash \bar b, a, \bar a \wedge b}}{\vdash \bar b, \bar b, b \wedge a, \bar a \wedge b}}{\vdash \bar b, b \wedge a, \bar a \wedge b}}{\vdash \bar b \wedge a, \bar a \wedge \bar b, b \wedge a, \bar a \wedge b}$$

$$\mathsf{s}\,\dfrac{\boxed{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{a \vee \bar a}} \wedge \boxed{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{b \vee \bar b}}}{\boxed{\mathsf{s}\,\dfrac{[a \vee \bar a] \wedge b}{(a \wedge b) \vee \bar a}} \vee \bar b}$$





This isn't even my final form

- We are still looking for an ideal language.
- We want to solve a fundamental problem: when are two proofs the same?
- They should be represented by the same object.
  - By giving simpler languages, we get closer to the answer.
- To build better languages, we need to understand how to better represent the properties that we like to study.

The coolest property

- The cut rule expresses that if we have a proof that A implies B and a proof that B implies C, then we have a proof that A implies C.
- We love systems that have the cut...
- ... but we want the cut to be "optional" or admissible.
  - for consistency
  - for proof search

The problem

- The process of showing that the cut is admissible is called cut-elimination.
- We study cut-elimination a lot!
  - Can it be done?
  - How much bigger are proofs without cuts?
- But it is very specific and very difficult, an important step when designing new proof systems...
- ... so I want to understand it.

THE shape

- To have a general cut-elimination, we need a general way to represent a proof system.
- Normally, rules have different shapes:

$$\frac{\mathsf{t}}{a \vee \bar{a}} \qquad\qquad \frac{(A \vee B) \wedge C}{(A \wedge C) \vee B}$$

- We present a shape to rule them all!

$$\frac{(A \; \alpha \; B) \; \beta \; (C \; \gamma \; D)}{(A \; \beta \; C) \; \alpha \; (B \; \delta \; D)}$$

In my thesis

- I show that many many many rules can be captured by THE shape.
- I divide cut-elimination into two different processes, and explain how each of them changes the size of a proof.
- I give general theorems for each of the two processes, and prove them.
- I give lots of examples of how the general theorems work when they are applied to specific systems.
- I give an idea of how some of the elements of the procedure are represented geometrically with our current language, and also reflect on how it is still possible to improve it by showing what we can't represent yet.

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