Generalising Cut-Elimination through Subatomic Proof Systems

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In work presented at PCC 2015 [1], we showed how we can consider atoms as self-dual, noncommutative binary logical relations. We can then define an interpretation map from formulae built in such a 'subatomic' way to ordinary formulae. Through this subatomic representation, we can see *all* the rules needed for a complete system for many propositional logics (including classical and linear ones) as instances of a single linear rule scheme

$$\frac{(A \, \alpha \, C) \, \beta \, (B \, \gamma \, D)}{(A \, \epsilon \, B) \, \zeta \, (C \, \eta \, D)} \quad ,$$

where the Greek letters denote logical connectives subject to certain simple conditions.

This ability to see *inside* of atoms gives us a way to reduce disparate rules such as contraction, cut, identity and any logical rule like conjunction-introduction, into a *unique* rule scheme. The fact that so many rules can be made to follow this rule scheme remains however quite surprising. It is an intriguing albeit very useful phenomenon, and we hope the audience to the talk can provide some insight as to the reason behind it. We can exploit having a single rule shape to reason generally on proof systems, allowing us to generalise methods that had to be proven to work for each individual system until now. This approach is particularly fruitful with respect to studying cut-elimination, and in particular it sheds light on *why* cut elimination works on such a wide range of proof systems. More concretely, by studying abstract systems where all the rules follow this scheme we are able to provide a general cut-elimination procedure for many systems without contraction, including all the standard variants of linear logic. Furthermore, by combining it with the decomposition techniques frequently used in deep inference, we can apply this cut-elimination method to an even wider range of systems, including Classical Logic. Looking at cut-elimination in this way helps us understand why it is such a prevalent phenomenon and why many similar cut-elimination arguments work in seemingly very different systems.

From the study of subatomic systems, we are able to give sufficient conditions for a system to enjoy cut-elimination. We provide a procedure, called splitting, that is a generalisation of a common technique employed for cut-elimination in deep inference systems. We show that splitting can be applied in many systems without contraction, such as Linear Logic [3] and including systems with self-dual non-commutative connectives such as BV [4]. The idea behind splitting is very simple, and it is rooted in deep inference methods. In the sequent calculus, formulae have a root connective that allows us to determine which rules are applied immediately above the cut and to follow a classical cut-elimination procedure by studying those rules. In deep inference, rules can be applied anywhere deep in a formula and as such anything can happen above a cut. As a consequence, the splitting method focuses on understanding the behaviour of the context around the cut, and in particular it consists on breaking down a proof in different pieces by following the logical connectives involved in the cut to find their duals. We show that we can then rearrange the different components of the proof to obtain a cut-free proof.

The reach of the splitting technique goes beyond linear systems when combined with another common deep inference method. In many systems, derivations can be arranged into consecutive subderivations made up of only certain rules [2,6]. We call this transformation method decomposition. A shining example of a decomposition theorem in the sequent calculus is Herbrand's Theorem, through which we can decompose a proof into a bottom phase with contraction and quantifier rules and a top phase with propositional rules only. Frequently, like in Classical Logic [5] , we can decompose a proof into a top phase without contractions, and a bottom phase made-up only of contractions. Furthermore, this decomposition is usually obtained by manipulating proofs through local transformations, rather than operating on the proof as a whole. By doing this, we can perform cut-elimination through splitting in the contraction-free top of the proof, and thus we can provide a cut-elimination procedure in two phases for all such decomposable systems.

By breaking down cut-elimination into these two different steps, we are able to observe hidden properties that cannot be explored with classical methods where decomposition and splitting are intertwined. For example, it allows us to study which parts of the cut-elimination process can be done locally, and which necessitate the manipulation of the proof as a whole. Interestingly, we can also observe that the complexity in this cut-elimination procedure stems from the decomposition methods rather than from the elimination of the cuts through splitting.

References

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