A study of normalisation through subatomic logic

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The observation

- We observe that deep inference systems have a recurring linear rule shape:

$$
ai\downarrow \frac{\mathsf{t}}{a \vee \bar{a}} \qquad ai\uparrow \frac{a \wedge \bar{a}}{\mathsf{f}}
$$

$$
s \frac{(A \vee B) \wedge C}{(A \wedge C) \vee B} \qquad m \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}
$$

$$
ac\downarrow \frac{a \vee a}{a} \qquad ac\uparrow \frac{a}{a \wedge a}
$$

$$
aw\downarrow \frac{\mathsf{f}}{a} \qquad aw\uparrow \frac{a}{\mathsf{t}}
$$

Figure: SKS [3]

$$
ai\downarrow \frac{1}{a \parr \bar{a}} \qquad ai\uparrow \frac{a \otimes \bar{a}}{\bot}
$$

$$
s \frac{(A \parr B) \otimes C}{(A \otimes C) \parr B}
$$

$$
d\downarrow \frac{(A \parr B) \mathbin{\&} (C \parr D)}{(A \mathbin{\&} C) \parr (B \oplus D)} \qquad d\uparrow \frac{(A \oplus B) \otimes (C \mathbin{\&} D)}{(A \otimes C) \oplus (B \otimes D)}
$$

$$
\oplus\downarrow \frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)} \qquad \&\uparrow \frac{(A \mathbin{\&} B) \otimes (C \mathbin{\&} D)}{(A \otimes C) \mathbin{\&} (B \otimes D)}
$$

$$
m \frac{(A \mathbin{\&} B) \oplus (C \mathbin{\&} D)}{(A \oplus C) \mathbin{\&} (B \oplus D)}
$$

$$
ac\downarrow \frac{a \oplus a}{a} \qquad ac\uparrow \frac{a}{a \mathbin{\&} a}
$$

$$
m_2\downarrow \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)} \qquad m_2\uparrow \frac{(A \mathbin{\&} B) \parr (C \mathbin{\&} D)}{(A \parr C) \mathbin{\&} (B \parr D)}
$$

$$
at\downarrow \frac{0}{a} \qquad at\uparrow \frac{a}{\top}
$$

Figure: SLLS [5]

One shape to rule them all

- Goal: generating propositional proofs by a single, linear, simple and regular inference rule scheme.

$$\frac{(A \; \alpha \; B) \; \beta \; (C \; \alpha' \; D)}{(A \; \beta \; C) \; \alpha \; (B \; \beta' \; D)}$$

One shape to rule them all

It turns out that atomic rules do follow this scheme.

- Idea: consider atoms as self-dual, noncommutative binary logical relations.

Subatomic systems

We use propositional classical logic as an example.

Idea: occurrences of an atom a are interpretations of more primitive expressions involving a noncommutative binary relation denoted by a.

- Formulae A and B in the relation a, in this order, are denoted by $A\ a\ B$.
- Formulae are built over the two units for disjunction and conjunction, respectively f and t.

Example: the following two expressions are SA formulae:

$$(\mathrm{f}\ a\ \mathrm{t}) \vee (\mathrm{t}\ a\ \mathrm{f}) \qquad (\mathrm{f}\ b\ \mathrm{t})\ a\ (\mathrm{t}\ c\ (\mathrm{t}\ d\ \mathrm{f})) \wedge \mathrm{f} \wedge ((\mathrm{f}\ a\ \mathrm{f}) \vee (\mathrm{t}\ b\ \mathrm{t}))$$

We call tame the formulae where atoms do not appear in the scope of other atoms (*e.g.*, left) and wild the others (*e.g.*, right).

The proof system

To interpret our extended language of formulae, we define an interpretation map $\mapsto$ from tame SA formulae to ordinary formulae such that

$$\begin{array}{ll} \mathsf{f}\ a\ \mathsf{t} \mapsto a & \mathsf{t}\ a\ \mathsf{f} \mapsto \bar{a} \\ \mathsf{t}\ a\ \mathsf{t} \mapsto \mathsf{t} & \mathsf{f}\ a\ \mathsf{f} \mapsto \mathsf{f} \end{array}$$

where $\bar{a}$ denotes the negation of a.
Note

- atoms are self dual: $\overline{A\ a\ B} \equiv \bar{A}\ a\ \bar{B}$
- atoms are not commutative
- atoms are not associative

We easily extend $\mapsto$ to all the tame SA formulae in the natural way.
For example: $(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{f}) \mapsto a \vee \bar{a}$ $\quad (\mathsf{f} \vee \mathsf{f})\ a\ (\mathsf{t} \vee \mathsf{t}) \mapsto a$

The proof system

Consider the usual contraction rule for an atom:

$$\frac{a \vee a}{a}$$

We could obtain this rule via $\mapsto$ as follows:

$$\frac{(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{f}\ a\ \mathsf{t})}{(\mathsf{f} \vee \mathsf{f})\ a\ (\mathsf{t} \vee \mathsf{t})} \mapsto \frac{a \vee a}{a} \qquad \text{and} \qquad \frac{(\mathsf{t}\ a\ \mathsf{f}) \vee (\mathsf{t}\ a\ \mathsf{f})}{(\mathsf{t} \vee \mathsf{t})\ a\ (\mathsf{f} \vee \mathsf{f})} \mapsto \frac{\bar{a} \vee \bar{a}}{\bar{a}} \quad .$$

We might consider those rules as generated by the linear scheme

$$\frac{(A\ a\ C) \vee (B\ a\ D)}{(A \vee B)\ a\ (C \vee D)}$$

This scheme is typical of logical rules in deep inference.

The proof system

Two more examples, identity and cut:

$$\frac{(\mathsf{f} \vee \mathsf{t})\ a\ (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{t}\ a\ \mathsf{f})} \mapsto \frac{\mathsf{t}}{a \vee \bar{a}} \qquad \text{and} \qquad \frac{(\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{t}\ a\ \mathsf{f})}{(\mathsf{f} \wedge \mathsf{t})\ a\ (\mathsf{t} \wedge \mathsf{f})} \mapsto \frac{a \wedge \bar{a}}{\mathsf{f}} \quad .$$

They are generated by the linear schemes:

$$\frac{(A \vee C)\ a\ (B \vee D)}{(A\ a\ B) \vee (C\ a\ D)} \qquad \text{and} \qquad \frac{(A\ a\ C) \wedge (B\ a\ D)}{(A \wedge B)\ a\ (C \wedge D)}$$

- Surprisingly, we are able to reduce disparate rules such as contraction, cut and identity into a unique rule scheme.

Doing proof theory

- A subatomic system SA is a deep inference system whose rules are instances of the inference rule scheme

$$\frac{(A\,\alpha\,B)\,\beta\,(C\,\alpha'\,D)}{(A\,\beta\,C)\,\alpha\,(B\,\beta'\,D)}$$

  where $\alpha' = \max(\alpha, \bar{\alpha})$ and $\beta' = \beta$, or, dually, $\beta' = \min(\beta, \bar{\beta})$ and $\alpha' = \alpha$.
  There are subatomic systems for CL, LL, BV, KV...
- A proof is a derivation whose premiss is t.
- A proof composed of only tame formulae corresponds to a proof in our usual proof theory.

Example: CL

$$\mathsf{a}{\downarrow}\ \frac{(A \vee B)\ \mathsf{a}\ (C \vee D)}{(A\ \mathsf{a}\ C) \vee (B\ \mathsf{a}\ D)} \qquad \mathsf{a}{\uparrow}\ \frac{(A\ \mathsf{a}\ B) \wedge (C\ \mathsf{a}\ D)}{(A \wedge C)\ \mathsf{a}\ (B \wedge D)}$$

$$\wedge{\downarrow}\ \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \qquad \vee{\uparrow}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$m\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\mathsf{ac}\ \frac{(A\ \mathsf{a}\ B) \vee (C\ \mathsf{a}\ D)}{(A \vee C)\ \mathsf{a}\ (B \vee D)} \qquad \mathsf{a}\bar{\mathsf{c}}\ \frac{(A \wedge B)\ \mathsf{a}\ (C \wedge D)}{(A\ \mathsf{a}\ C) \wedge (B\ \mathsf{a}\ D)}$$

Figure: SAKS [1]

Example: MALL

$$a{\downarrow}\ \frac{(A \parr B)\ a\ (C \parr D)}{(A\ a\ C) \parr (B\ a\ D)} \qquad a{\uparrow}\ \frac{(A\ a\ B) \otimes (C\ a\ D)}{(A \otimes C)\ a\ (B \otimes D)}$$

$$\otimes{\downarrow}\ \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr{\uparrow}\ \frac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)}$$

$$\&{\downarrow}\ \frac{(A \parr B) \,\&\, (C \parr D)}{(A \,\&\, C) \parr (B \oplus D)} \qquad \oplus{\uparrow}\ \frac{(A \oplus B) \otimes (C \,\&\, D)}{(A \otimes C) \oplus (B \otimes D)}$$

$$\oplus{\downarrow}\ \frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)} \qquad \&{\uparrow}\ \frac{(A \,\&\, B) \otimes (C \,\&\, D)}{(A \otimes C) \,\&\, (B \otimes D)}$$

$$\parr{\downarrow}\ \frac{(A \parr B) \parr (C \parr D)}{(A \parr C) \parr (B \parr D)} \qquad \otimes{\uparrow}\ \frac{(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$m\ \frac{(A \,\&\, B) \oplus (C \,\&\, D)}{(A \oplus C) \,\&\, (B \oplus D)}$$

$$ac{\downarrow}\ \frac{(A\ a\ B) \oplus (C\ a\ D)}{(A \oplus C)\ a\ (B \oplus D)} \qquad ac{\uparrow}\ \frac{(A \,\&\, B)\ a\ (C \,\&\, D)}{(A\ a\ C) \,\&\, (B\ a\ D)}$$

$$\otimes c{\downarrow}\ \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)} \qquad \parr c{\uparrow}\ \frac{(A \,\&\, B) \parr (C \,\&\, D)}{(A \parr C) \,\&\, (B \parr D)}$$

$$\oplus c{\downarrow}\ \frac{(A \oplus B) \oplus (C \oplus D)}{(A \oplus C) \oplus (B \oplus D)} \qquad \&c{\uparrow}\ \frac{(A \,\&\, B) \,\&\, (C \,\&\, D)}{(A \,\&\, C) \,\&\, (B \,\&\, D)}$$

Figure: SAMALLS [1]

Splitting

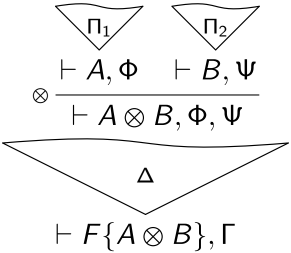
- We can characterise splittable systems [2] for which cut-elimination is ensured.
    1. There is a distinguished connective $+$ with unit 0.
    2. All rules are of the form

$$\alpha\downarrow \frac{(A + B)\,\alpha\,(C + D)}{(A\,\alpha\,C) + (B\,\alpha^m\,D)} \quad .$$

- They correspond to a class of substructural logics: those without contractions.

Splitting

- We exploit the fact that we can always find independent proofs above a cut.



Splitting

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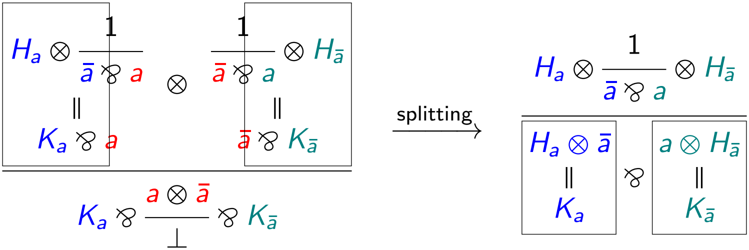


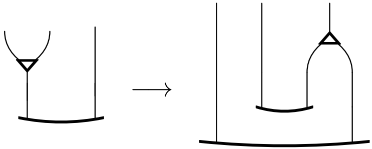
Splitting

- Splitting goes beyond cut-elimination: we can show the admissibility of a family of rules.
- Global procedure of polynomial-time complexity.

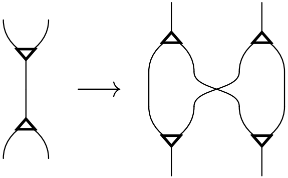
Decomposition

In some deep inference systems, we can permute atomic contractions to the bottom of proofs through local reductions [4].

$$\frac{\boxed{\dfrac{a \vee a}{a}} \wedge \bar{a}}{\mathsf{f}} \quad\longrightarrow\quad \frac{(a \vee a) \wedge \boxed{\dfrac{\bar{a}}{(\bar{a} \wedge \bar{a})}}}{\dfrac{\boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}}}{\mathsf{f}}}$$



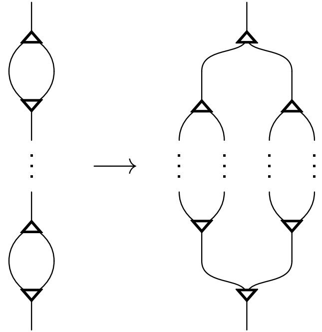
$$\frac{\dfrac{a \vee a}{a}}{a \wedge a} \quad\longrightarrow\quad \frac{\boxed{\dfrac{a}{a \wedge a}} \vee \boxed{\dfrac{a}{a \wedge a}}}{\boxed{\dfrac{a \vee a}{a}} \wedge \boxed{\dfrac{a \vee a}{a}}}$$



Decomposition

We can pinpoint exactly where an exponential increase on the size of proofs occurs.

$$\cfrac{\cfrac{a \vee a}{a}}{a \wedge a} \longrightarrow \cfrac{\boxed{\cfrac{a}{a \wedge a}} \vee \boxed{\cfrac{a}{a \wedge a}}}{\boxed{\cfrac{a \vee a}{a}} \wedge \boxed{\cfrac{a \vee a}{a}}}$$



Decomposition

We can generalise contractions and give conditions for reduction rules to hold:

- If for the relation β there is a rule $\dfrac{(A \vee B)\,\beta\,(C \wedge D)}{(A\,\beta\,C) \vee (B\,\beta\,D)}$, the following reduction holds:

$$\frac{\boxed{\text{mc}{\downarrow}\,\dfrac{(A\;\alpha\;B)\vee(C\;\alpha\;D)}{(A\vee C)\;\alpha\;(B\vee D)}}\;\beta\;(E\;\alpha'\;F)}{((A\vee C)\;\beta\;E)\;\alpha\;((B\vee D)\;\beta'\;F)} \longrightarrow \text{mc}{\downarrow}\,\frac{\dfrac{((A\;\alpha\;B)\vee(C\;\alpha\;D))\;\beta\;\boxed{\text{mc}{\uparrow}\,\dfrac{E\;\alpha'\;F}{(E\;\alpha'\;F)\wedge(E\;\alpha'\;F)}}}{\boxed{\dfrac{(A\;\alpha\;B)\;\beta\;(E\;\alpha'\;F)}{(A\;\beta\;E)\;\alpha\;(B\;\beta'\;F)}}\vee\boxed{\dfrac{(C\;\alpha\;D)\;\beta\;(E\;\alpha'\;F)}{(C\;\beta\;E)\;\alpha\;(D\;\beta'\;F)}}}}{((A\vee C)\;\beta\;E)\;\alpha\;((B\vee D)\;\beta'\;F)}$$

Decomposition

$$\frac{\boxed{\mathsf{mc}\!\downarrow\frac{(A\,\alpha\,B)\vee(C\,\alpha\,D)}{(A\vee C)\,\alpha\,(B\vee D)}}\ \beta\ (E\,\alpha'\,F)}{((A\vee C)\,\beta\,E)\,\alpha\,((B\vee D)\,\beta'\,F)} \longrightarrow \frac{((A\,\alpha\,B)\vee(C\,\alpha\,D))\,\beta\,\boxed{\mathsf{mc}\!\uparrow\frac{E\,\alpha'\,F}{(E\,\alpha'\,F)\wedge(E\,\alpha'\,F)}}}{\mathsf{mc}\!\downarrow\dfrac{\boxed{\frac{(A\,\alpha\,B)\,\beta\,(E\,\alpha'\,F)}{(A\,\beta\,E)\,\alpha\,(B\,\beta'\,F)}}\vee\boxed{\frac{(C\,\alpha\,D)\,\beta\,(E\,\alpha'\,F)}{(C\,\beta\,E)\,\alpha\,(D\,\beta'\,F)}}}{((A\vee C)\,\beta\,E)\,\alpha\,((B\vee D)\,\beta'\,F)}}$$

$$\frac{\boxed{\frac{a\vee a}{a}}\wedge\bar a}{\mathsf{f}} \longrightarrow \frac{(a\vee a)\wedge\boxed{\frac{\bar a}{(\bar a\wedge\bar a)}}}{\dfrac{\boxed{\frac{a\wedge\bar a}{\mathsf{f}}}\vee\boxed{\frac{a\wedge\bar a}{\mathsf{f}}}}{\mathsf{f}}}$$

Decomposition

- The behaviour of atomic contractions is a particular case of a more generalised behaviour.
- Local procedure of exponential complexity.

Conclusion

- We observe a mysterious phenomenon: only one rule shape is enough to describe many different systems.
- We exploit it to reason generally and untangle two different interactions involving cut-elimination.
- We can exploit it to design systems.
- Towards braids.

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